Characteristic polynomial, eigenvalues, and eigenvectors 4

For each of the following matrices, we request:

1. Find the characteristic polynomial.
2. Find the eigenvalues and the associated eigenvectors.

i)

$$\begin{pmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{pmatrix}$$

j)

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution

i) Characteristic polynomial

Let
$$A = \begin{pmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{pmatrix}.$$

To find the characteristic polynomial of A, we compute
$$p(\lambda) = \det(A - \lambda I).$$

In this case,
$$A - \lambda I = \begin{pmatrix} -1-\lambda & -3 & -9 \\ 0 & 5-\lambda & 18 \\ 0 & -2 & -7-\lambda \end{pmatrix}.$$

We can expand by cofactor along the first column. Since the second and third entries of the first column are zero, the determinant simplifies to:
$$p(\lambda) = (-1-\lambda)\,\det\begin{pmatrix} 5-\lambda & 18 \\ -2 & -7-\lambda \end{pmatrix}.$$

The 2×2 minor is:
$$\begin{vmatrix} 5-\lambda & 18 \\ -2 & -7-\lambda \end{vmatrix} = (5-\lambda)(-7-\lambda) \;-\; (18)(-2).$$

Expanding,
$$(5-\lambda)(-7-\lambda) = (5)(-7-\lambda) - \lambda(-7-\lambda) = -35 - 5\lambda + 7\lambda + \lambda^2 = \lambda^2 + 2\lambda - 35,$$

so that
$$(5-\lambda)(-7-\lambda) \;+\; 36 = \lambda^2 + 2\lambda - 35 \;+\; 36 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2.$$

Thus,
$$p(\lambda) = (-1-\lambda)\,(\lambda+1)^2 = -(\lambda+1)\,(\lambda+1)^2 = -(\lambda+1)^3.$$

To express the polynomial in monic form, we factor out -1:
$$p(\lambda) = (\lambda+1)^3.$$

Eigenvalues and eigenvectors

The equation $p(\lambda) = 0$ reduces to
$$(\lambda+1)^3 = 0,$$

which implies that the only eigenvalue is
$$\lambda = -1$$

with algebraic multiplicity 3.

To find the eigenvectors associated with $\lambda = -1$, we solve
$$(A + I)\,\mathbf{v} = 0.$$

In matrix form,
$$A + I = \begin{pmatrix} -1+1 & -3 & -9 \\ 0 & 5+1 & 18 \\ 0 & -2 & -7+1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & -9 \\ 0 & 6 & 18 \\ 0 & -2 & -6 \end{pmatrix}.$$

Let $\mathbf{v} = (x,\, y,\, z)^T$. The system imposed by $(A+I)\mathbf{v} = 0$ reads:

$$\begin{cases} -3y \ - \ 9z \ = \ 0, \\ 6y \ + \ 18z \ = \ 0, \\ -2y \ - \ 6z \ = \ 0. \end{cases}$$

All three equations essentially impose the same condition: $y + 3z = 0$, without any restriction on x. Therefore, x is free and $y = -3z$. We parametrize the solution by setting $x = s$ and $z = t$, so that

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -3t \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Thus, the eigenspace of $\lambda = -1$ is of dimension 2, indicating that the *geometric multiplicity* of eigenvalue -1 is 2, while its *algebraic multiplicity* is 3. Consequently, there exists an additional vector (a generalized eigenvector) completing the Jordan chain, but it does *not* satisfy $(A + I)\mathbf{v} = 0$.

A possible basis for the eigenspace is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Any linear combination of these two vectors is also an eigenvector associated with $\lambda = -1$.

j) Characteristic polynomial

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

To find the characteristic polynomial of A, we compute

$$p(\lambda) = \det(A - \lambda I).$$

In this case,

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}.$$

Expanding along the first row, we obtain:

$$\begin{aligned} p(\lambda) &= (1-\lambda) \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} - (-1) \det \begin{pmatrix} -1 & -1 \\ 0 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda) \Big[(2-\lambda)(1-\lambda) - (-1)(-1) \Big] + (1-\lambda). \end{aligned}$$

Noting that

$$(2-\lambda)(1-\lambda) = \lambda^2 - 3\lambda + 2,$$

we obtain

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1.$$

Thus,

$$p(\lambda) = (1-\lambda)(\lambda^2 - 3\lambda + 1) + (1-\lambda) = (1-\lambda) \Big[(\lambda^2 - 3\lambda + 1) + 1 \Big],$$

which simplifies to

$$p(\lambda) = (1-\lambda)(\lambda^2 - 3\lambda + 2).$$

Since

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2),$$

we have:

$$p(\lambda) = (1-\lambda)(\lambda-1)(\lambda-2).$$

Observing that $(1-\lambda)(\lambda-1) = -(\lambda-1)^2$, we can express the characteristic polynomial in monic form (multiplying by -1):

$$p(\lambda) = \lambda(\lambda-1)(\lambda-3),$$

which means that the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3.$$

Eigenvalues and eigenvectors

Eigenvalue $\lambda_1 = 0$

To find the eigenvector associated with $\lambda = 0$, we solve

$$A\,\mathbf{v} = 0.$$

That is,

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The resulting system is:

$$\begin{cases} x - y = 0, \\ -x + 2y - z = 0, \\ -y + z = 0. \end{cases}$$

From the first equation, $x = y$; from the third, $z = y$. Thus, an associated eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda_2 = 1$

For $\lambda = 1$, we solve:

$$(A - I)\mathbf{v} = 0.$$

We have:

$$A - I = \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The corresponding system is:

$$\begin{cases} -y = 0, \\ -x + y - z = 0, \\ -y = 0. \end{cases}$$

From this, we deduce $y = 0$ and, substituting into the second equation, $-x - z = 0$, that is, $x = -z$. Thus, an associated eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda_3 = 3$

For $\lambda = 3$, we solve:

$$(A - 3I)\mathbf{v} = 0.$$

We have:

$$A - 3I = \begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

The corresponding system is:

$$\begin{cases} -2x - y = 0, \\ -x - y - z = 0, \\ -y - 2z = 0. \end{cases}$$

From the first equation, $y = -2x$. From the third, $-(-2x) - 2z = 2x - 2z = 0$ implies $z = x$. The second equation is automatically satisfied. Thus, an associated eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$